

The Spend-It-All Region and Small Time Results for the Continuous Bomber Problem

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Abstract: A problem of optimally allocating partially effective ammunition x to be used on randomly arriving enemies in order to maximize an aircraft's probability of surviving for time t , known as the Bomber Problem, was first posed by Klinger and Brown (1968). They conjectured a set of apparently obvious monotonicity properties of the optimal allocation function $K(x, t)$. Although some of these conjectures, and versions thereof, have been proved or disproved by other authors since then, the remaining central question, that $K(x, t)$ is nondecreasing in x , remains unsettled. After reviewing the problem and summarizing the state of these conjectures, in the setting where x is continuous we prove the existence of a “spend-it-all” region in which $K(x, t) = x$ and find its boundary, inside of which the long-standing, unproven conjecture of monotonicity of $K(\cdot, t)$ holds. A new approach is then taken of directly estimating $K(x, t)$ for small t , providing a complete small- t asymptotic description of $K(x, t)$ and the optimal probability of survival.

Keywords: Ammunition rationing; Optimal allocation; Poisson process; Sequential optimization.

Subject Classifications: 60G40; 62L05; 91A60.

1. INTRODUCTION

Klinger and Brown (1968) introduced a problem of optimally allocating partially effective ammunition to be used on enemies arriving at a Poisson rate in order to maximize the probability that an aircraft (hereafter “the bomber”) survives for time t , known as the Bomber Problem. Given an amount x of ammunition, let $K(x, t)$ denote the optimal amount of ammunition the bomber would use upon confronting

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an enemy at *time* t , defined as the time remaining to survive. The appearance of enemies is driven by a time-homogeneous Poisson process of known rate, taken to be 1. An enemy survives the bomber's expenditure of an amount $y \in [0, x]$ of its ammunition with the geometric probability q^y , for some known $q \in (0, 1)$, after which the enemy has a chance to destroy the bomber, which happens with known probability $v \in (0, 1]$ (the $v = 0$ case being trivial). By rescaling x , we assume without loss of generality that $q = e^{-1}$, and hence the probability that the bomber survives an enemy encounter in which it spends an amount y of its ammunition is

$$a(y) = 1 - ve^{-y}. \quad (1.1)$$

Klinger and Brown (1968) posed two seemingly obvious conjectures about the optimal allocation function $K(x, t)$:

- A: $K(x, t)$ is nonincreasing in t for all fixed $x \geq 0$;
 B: $K(x, t)$ is nondecreasing in x for all fixed $t \geq 0$.

Klinger and Brown (1968) showed that [B] implies [A] when $v = 1$, although, as will be discussed below, [B] remains in doubt. Improving the situation, Samuel (1970) showed that [A] holds without assuming [B] in the setting where units of ammunition x are discrete, and in this setting also showed that a third conjecture holds:

- C: The amount $x - K(x, t)$ held back by the bomber is nondecreasing in x for all fixed $t \geq 0$.

[C] was first stated as a formal property by Simons and Yao (1990), who claimed that it can be shown to hold for continuous x and t by arguments similar to the ones they provide for a case where both x and t are discrete, and they also make theoretical and computational progress toward [B] in various discrete/continuous settings. Also in the setting where both x and t are continuous, Bartroff et al. (2010) recently showed that [A] holds, and provide a full proof of [C] in this setting. Weber (1985) considered an infinite-horizon variant of the Bomber Problem in which the objective is to maximize the number of enemies shot down (thus removing t from the problem) and found that, for discrete x , the property related to [B], that of monotonicity of $K(x)$, fails to hold. Shepp et al. (1991) considered the infinite-horizon problem for continuous x and reached the same conclusion. On the other hand, Bartroff et al. (2010) consider the variation of the problem where the bomber is invincible, and both x and t are present and continuous, and show that [B] holds.

In spite of the results of Weber (1985), Shepp et al. (1991), and Bartroff et al. (2010), conjecture [B] has not been settled in any close relative to the original Bomber Problem, and it remains the conjecture about which the least is known. To gain insight into the function $K(x, t)$, perhaps as a step towards resolving [B] in greater generality, we take a new approach to the Bomber Problem of directly estimating, or when possible solving for, $K(x, t)$ when both x and t are continuous. One might expect a priori that if x or t is sufficiently small then the optimal strategy is to spend all or nearly all of the available ammunition x , i.e., that $K(x, t)$ is equal to or nearly x . On the other hand, since the ammunition is assumed to be continuous it is not obvious that there exists a "spend-it-all" region where $K(x, t)$ is *identically* x . In Section 2 we show that there is indeed a spend-it-all region of (x, t) values for

which $K(x, t) = x$ and where [B] holds, and we estimate the region’s boundary in Theorem 2.1, and are able to find it exactly in most cases. However, in Section 3 we show that there are many other regimes in which $K(x, t)$ is not so simple, but can nevertheless be described asymptotically for small values of t . In particular, in Theorem 3.1 we characterize the asymptotic behavior of $K(x, t)$ for small t and show that regardless of how small t is, there are large intervals of x values for which $K(x, t)/x$ approaches any, even arbitrarily small, positive fraction, in stark contrast to the spend-it-all strategy. The relation of these results to the outstanding conjecture [B] and extensions are discussed in Section 4.

2. THE SPEND-IT-ALL REGION

In this section we describe an (x, t) -region where $K(x, t)$ is identically x , the so-called “spend-it-all” region. The boundary of this region is solved for, exactly as (2.1), except for a special configuration of the parameters x, t, v in which the boundary is estimated from both sides; see (2.8). Bartroff (2010) has in the meantime shown that (2.1) always gives the exact boundary of the spend-it-all region.

In what follows, let $u = 1 - v \in [0, 1)$ denote the probability that the bomber survives an enemy’s counterattack, let $P(x, t)$ denote the optimal probability of survival at time t when the bomber has ammunition x , and let $H(x, t)$ denote the optimal conditional probability of survival given an enemy at time t , with ammunition x .

Theorem 2.1. *For $u \in (0, 1)$ and $t > 0$ define*

$$f_u(t) = \log[1 + u/(e^{tu} - 1)], \tag{2.1}$$

and extend this definition to $u = 0$ by defining

$$f_0(t) = \lim_{u \rightarrow 0} f_u(t) = \log(1 + t^{-1}).$$

For $u \in [0, 1)$ and $t > 0$ define

$$g_u(t) = \log(1 + t^{-1} - u). \tag{2.2}$$

If $u \in [0, 1)$ and $t > 0$ satisfy one of the following:

(i) $u = 0,$ (2.3)

(ii) $u \in (0, 1/2)$ and $t \geq u^{-1} \log(2v),$ (2.4)

(iii) $u \in [1/2, 1),$ (2.5)

then

$$K(x, t) = x \text{ if and only if } x \leq f_u(t). \tag{2.6}$$

In the remaining case, where

$$u \in (0, 1/2) \text{ and } t < u^{-1} \log(2v), \tag{2.7}$$

we have

$$K(x, t) = x \text{ if } x \leq g_u(t), \text{ and } K(x, t) < x \text{ if } x > f_u(t). \tag{2.8}$$

The theorem may be summarized by saying that, except for the configuration of t, u values in (2.7), the spend-it-all region's boundary is given exactly by $f_u(t)$, which is positive for all $t > 0$ and approaches 0 as $t \rightarrow \infty$. Although the authors conjecture that $f_u(t)$ is the boundary of the spend-it-all region for all $t > 0$ and $u \in [0, 1)$, this is not shown here in the remaining case (2.7). Instead, in this case the boundary is estimated from above by $f_u(t)$ and from below by $g_u(t)$, which is strictly less than $f_u(t)$ for all $t > 0$ but asymptotically equivalent to it as $t \rightarrow 0$. Although $g_u(t)$ is negative for $t > u^{-1}$, it is utilized as a bound only when (2.7) holds, in which case $u^{-1} > u^{-1} \log(2v) > 0$. A consequence of the theorem is that, regardless of the value of u , for any $x > 0$ there is t sufficiently small such that the optimal strategy spends it all (i.e., $K(x, t) = x$), and for any $t > 0$ there is x sufficiently small such that the optimal strategy spends it all.

Proof. We first prove that $K(x, t) = x$ when x is bounded from above by $f_u(t)$ and one of (2.3)–(2.5) holds, or when x is bounded from above by $g_u(t)$ and (2.7) holds. To begin, fix x, t and let u be any value in $[0, 1)$. We make use of the crude upper bound on the optimal survival probability

$$P(x, t) \leq \exp(-vte^{-x}) \text{ for all } x, t > 0, \tag{2.9}$$

which corresponds to the infeasible strategy of firing an amount x of ammunition at every possible enemy, giving

$$P(x, t) \leq \sum_{i=0}^{\infty} e^{-t} [ta(x)]^i / i! = e^{-t} e^{ta(x)} = e^{-t(1-a(x))} = \exp(-vte^{-x}).$$

Using (2.9), the optimal conditional survival probability is then

$$H(x, t) = a(K(x, t))P(x - K(x, t), t) \leq F(x - K(x, t)),$$

where for fixed x and t we write

$$F(y) = a(x - y) \exp(-vte^{-y}).$$

By Lemma 2.1 below, F is unimodal on \mathbb{R} with maximum at

$$y^* = \log \left(-vt + \sqrt{v^2t^2 + 4te^x} \right) - \log 2.$$

which is not necessarily in $[0, x]$. In fact, if $x \leq g_u(t)$, then

$$\begin{aligned} y^* &\leq \log \left(-vt + \sqrt{v^2t^2 + 4te^{g_u(t)}} \right) - \log 2 \\ &= \log \left(-vt + \sqrt{v^2t^2 + 4t(1 + t^{-1} - u)} \right) - \log 2 \\ &= \log \left(-vt + \sqrt{(vt + 2)^2} \right) - \log 2 \\ &= 0, \end{aligned}$$

hence in this case $\max_{y \in [0, x]} F(y) = F(0) = a(x)e^{-tv}$. If it were that $K(x, t) < x$, then we would have

$$H(x, t) \leq F(x - K(x, t)) < F(0) = a(x)e^{-tv}, \quad (2.10)$$

a contradiction because the latter is the conditional survival probability of the spend-it-all strategy:

$$a(x) \sum_{i=0}^{\infty} u^i e^{-t} t^i / i! = a(x) e^{-t} e^{tu} = a(x) e^{-tv}. \quad (2.11)$$

Note that e^{-tv} is the probability of not being killed in the enemy's thinned Poisson process with parameter v . The argument leading to (2.10) thus shows that $K(x, t) = x$ whenever $x \leq g_u(t)$; in particular, $K(x, t) = x$ when (2.7) holds, or when (2.3) holds after noting that $g_0(t) = f_0(t)$. For the remaining cases (2.4) and (2.5), we obtain a tighter bound. Fix x, t and let $u \in (0, 1)$. Letting

$$G(y) = a(x - y)e^{-t} [1 + e^{vy/u}(e^{tu} - 1)],$$

we claim that

$$H(x, t) \leq G(x - K(x, t)). \quad (2.12)$$

To prove (2.12), first, a simple verification yields that for any nonnegative b_1, \dots, b_i ,

$$\prod_{j=1}^i a(b_j) \leq a(y/i)^i \quad \text{when} \quad \sum_{j=1}^i b_j = y. \quad (2.13)$$

Hence, $H(x, t) \leq \tilde{G}(x - K(x, t))$, where

$$\tilde{G}(y) = a(x - y)e^{-t} \left[1 + \sum_{i=1}^{\infty} \frac{(ta(y/i))^i}{i!} \right],$$

as the right hand side is the probability of survival for the infeasible strategy where one is given the number i of future encounters, and divides the remaining amount $x - K(x, t)$ of ammunition optimally among them, firing $(x - K(x, t))/i$ at each. Next, we claim that

$$a(y/i)^i \leq u^i e^{vy/u} \quad \text{for all } y \in [0, x] \text{ and all } i \geq 1, \quad (2.14)$$

implying that $\tilde{G}(y) \leq G(y)$ for all $y \in [0, x]$, and hence (2.12). Letting $\rho_i = [a(y/i)/u]^i$, (2.14) is true since $\lim_{i \rightarrow \infty} \rho_i = e^{vy/u}$ and ρ_i is evidently a nondecreasing sequence:

$$\begin{aligned} u^i(\rho_i - \rho_{i-1}) &= a(y/i)^i - ua(y/(i-1))^{i-1} \\ &= a(y/i)^i - a(0)a(y/(i-1))^{i-1}, \\ &\geq 0, \end{aligned}$$

this last by (2.13). We will show below that if (2.4) or (2.5) holds, then $G(y)$ is uniquely maximized over $y \in [0, x]$ at $y = 0$. Since $G(0) = a(x)e^{-tv}$, it then follows that $K(x, t) = x$, as above. To verify the maximum of G , we show that $G'(0) \leq 0$ and $G''(y) < 0$ for all $y \in (0, x]$. We compute

$$\begin{aligned} e^t G'(y) &= -\frac{v}{u} \{e^{-x}[ue^y + e^{y/u}(e^{tu} - 1)] - e^{vy/u}(e^{tu} - 1)\} \\ e^t G''(y) &= -\frac{v}{u^2} \{e^{-x}[u^2 e^y + e^{y/u}(e^{tu} - 1)] - ve^{vy/u}(e^{tu} - 1)\}. \end{aligned}$$

If $x \leq f_u(t)$, which is equivalent to $e^{-x} \geq (1 + u/(e^{tu} - 1))^{-1}$, then we have

$$\begin{aligned} -\left(\frac{u}{v}\right)e^t G'(0) &= e^{-x}(u + (e^{tu} - 1)) - (e^{tu} - 1) \\ &\geq \left(1 + \frac{u}{e^{tu} - 1}\right)^{-1} (e^{tu} - v) - (e^{tu} - 1) \\ &= \left(\frac{e^{tu} - 1}{e^{tu} - v}\right)(e^{tu} - v) - (e^{tu} - 1) \\ &= 0, \end{aligned}$$

hence $G'(0) \leq 0$. Next,

$$\begin{aligned} -\left(\frac{u^2 e^{-vy/u}}{v}\right)e^t G''(y) &= e^{-x}[u^2 e^{(2u-1)y/u} + e^y(e^{tu} - 1)] - v(e^{tu} - 1) \\ &= e^{-x}p(y) - v(e^{tu} - 1), \end{aligned}$$

where $p(y) = u^2 e^{(2u-1)y/u} + e^y(e^{tu} - 1)$. When $u \geq 1/2$ the function $p(y)$ is clearly increasing in y so for $y > 0$ and $x \leq f_u(t)$,

$$\begin{aligned} -\left(\frac{u^2 e^{-vy/u}}{v}\right)e^t G''(y) &> e^{-x}p(0) - v(e^{tu} - 1) \\ &= e^{-x}(u^2 + e^{tu} - 1) - v(e^{tu} - 1) \\ &\geq \left(\frac{e^{tu} - 1}{e^{tu} - v}\right)(u^2 + e^{tu} - 1) - v(e^{tu} - 1) \\ &= \left(\frac{e^{tu} - 1}{e^{tu} - v}\right)[u^2 + e^{tu} - 1 - v(e^{tu} - v)] \\ &= \left(\frac{e^{tu} - 1}{e^{tu} - v}\right)[u(e^{tu} - 2u)] \\ &\geq 0, \end{aligned} \tag{2.15}$$

since $u \geq 1/2$ implies that $2v \leq 1 \leq e^{tu}$. Finally, we show that when (2.4) holds, $p(y)$ is still increasing. First compute

$$\begin{aligned} p'(y) &= u(2u - 1)e^{(2u-1)y/u} + e^y(e^{tu} - 1), \\ p''(y) &= (2u - 1)^2 e^{(2u-1)y/u} + e^y(e^{tu} - 1) > 0, \end{aligned}$$

and

$$\begin{aligned}
 p'(0) &= u(2u - 1) + (e^{tu} - 1) \\
 &\geq u(2u - 1) + (2v - 1) \quad (\text{since } t \geq u^{-1} \log(2v)) \\
 &= 2u^2 - 3u + 1 \\
 &= 2(1 - u)(1/2 - u) \\
 &> 0
 \end{aligned}$$

since $u < 1/2$. Thus, the steps leading to (2.15) hold in this case as well, completing the proof that $K(x, t) = x$ when (2.3), (2.4), (2.5), or (2.7) holds.

To complete the proof of the theorem, we show that $K(x, t) < x$ when $x > f_u(t)$. To do this, we bound $H(x, t)$ from below by the conditional survival probability $\underline{H}(y)$ of the strategy that fires an amount $y \in [0, x]$ of ammunition at the present enemy, fires all remaining ammunition $x - y$ at the next enemy (if one is encountered), and hopes for the best thereafter. First assume that $u \in (0, 1)$ and fix x, t satisfying $x > f_u(t)$. Then

$$\begin{aligned}
 \underline{H}(y) &= a(y) \left[e^{-t} + e^{-t} \sum_{i=1}^{\infty} \frac{t^i a(x-y) u^{i-1}}{i!} \right] \\
 &= a(y) \left[e^{-t} + e^{-t} \frac{a(x-y)}{u} (e^{tu} - 1) \right] \\
 &= e^{-t} a(y) \left[1 + \left(\frac{e^{tu} - 1}{u} \right) a(x-y) \right].
 \end{aligned}$$

By applying Lemma 2.1 with $A = (e^{tu} - 1)/u$, we see that $\underline{H}(y)$ is unimodal with maximum at $K^*(x, t) = (x + f_u(t))/2$, which, since $x > f_u(t)$, satisfies $K^*(x, t) < (x + x)/2 = x$. If it were that $K(x, t) = x$, then we would have

$$H(x, t) = a(x)e^{-tv} = \underline{H}(x) < \underline{H}(K^*(x, t)),$$

a contradiction. If $u = 0$, the conditional survival probability of this strategy is

$$\underline{H}(y) = a(y)[e^{-t} + e^{-t}ta(x-y)] = e^{-t}a(y)[1 + ta(x-y)],$$

and a similar argument applies: By Lemma 2.1 with $A = t$, the function $\underline{H}(y)$ is unimodal with maximum at $K^*(x, t) = (x + f_0(t))/2 < x$, leading to the same contradiction. \square

Lemma 2.1. Fix $x > 0$, $t > 0$, and $v \in (0, 1]$. The function

$$y \mapsto a(x-y) \exp(-vte^{-y}) \tag{2.16}$$

is unimodal on \mathbb{R} with maximum at

$$y^* = \log \left(-vt + \sqrt{v^2t^2 + 4te^x} \right) - \log 2. \tag{2.17}$$

For any fixed $A > 0$, the function

$$y \mapsto a(y)[1 + Aa(x - y)] \tag{2.18}$$

is unimodal on \mathbb{R} with maximum at

$$y^* = [x + \log(1 + A^{-1})]/2. \tag{2.19}$$

Proof. Taking the derivative of (2.16) with respect to y and setting $z = e^y$ gives

$$-ve^{-x-y} \exp(-vte^{-y})(e^{2y} + vte^y - te^x) = -ve^{-x-y} \exp(-vte^{-y})(z^2 + vtz - te^x). \tag{2.20}$$

Since $z > 0$, the function (2.16) increases in $y = \log z$ up to the log of the positive root of the quadratic in (2.20), which is (2.17), and decreases thereafter. Similarly, the derivative of (2.18) with respect to y is

$$-v(Ae^{-x+y} - (1 + A)e^{-y}) = -ve^{-y}(Ae^{-x}z^2 - (1 + A)),$$

and solving for the root gives (2.19). □

3. AN ASYMPTOTIC CHARACTERIZATION OF $K(x, t)$

In this section we give an asymptotic description of the optimal allocation function $K(x, t)$ as $t \rightarrow 0$, and for this it suffices to consider sequences (x, t) with $t \rightarrow 0$. In addition to giving an asymptotic description of the optimal survival probability $P(x, t)$ and the optimal conditional survival probability $H(x, t)$, our main goal is to characterize the fraction $K(x, t)/x$ of the current ammunition x spent by the optimal strategy at time t , and it turns out that $K(x, t)/x$ approaches a finite nonzero limit on sequences (x, t) such that $|\log t|/x$ approaches a finite nonzero limit. We thus give an essentially complete asymptotic description of $K(x, t)$ by considering sequences $(x, t) = (x_t, t)$ such that

$$\frac{|\log t|}{x} \rightarrow \rho \in (0, \infty) \text{ as } t \rightarrow 0, \tag{3.1}$$

leaving divergent sequences to be handled by considering subsequences. We will write $x = x_t$ when we wish to emphasize the dependence of x on t , but most of the time this notation will be suppressed. Note that a consequence of (3.1) is that $x \rightarrow \infty$ at the same rate at which $|\log t| \rightarrow \infty$ as $t \rightarrow 0$. It should perhaps not be surprising that this is the nontrivial asymptotic regime since the boundary of the spend-it-all region found in Theorem 2.1 is asymptotically equivalent to $|\log t|$ as $t \rightarrow 0$. In what follows, let $\binom{1}{2}^{-1} = \infty$.

Theorem 3.1. *Under (3.1), let $j \in \{1, 2, \dots\}$ be such that*

$$\binom{j+1}{2}^{-1} \leq \rho < \binom{j}{2}^{-1}. \tag{3.2}$$

Then, as $t \rightarrow 0$,

$$\frac{K(x, t)}{x} \rightarrow 1/j + \rho(j-1)/2 \quad (3.3)$$

$$\frac{1}{x} |\log(1 - H(x, t))| \rightarrow 1/j + \rho(j-1)/2 \quad (3.4)$$

$$\frac{1}{x} |\log(1 - P(x, t))| \rightarrow 1/j + \rho(j+1)/2. \quad (3.5)$$

The theorem is proved in the next subsection. First, we briefly discuss the result. Note that the j satisfying (3.2) is nonincreasing in ρ and, in particular, $\rho \geq 1$ corresponds to $j = 1$ whereas $\rho < 1$ corresponds to $j > 1$. The right hand sides of (3.3) and (3.4) equal 1 for $j = 1$, and are in the interval $[2/(j+1), 2/j]$ for $j \geq 2$; similarly, the right hand side of (3.5) is in the interval $[2/j, 2/(j-1)]$ for all $j \geq 1$. In particular, (3.3) implies that $K(x, t)/x$ can take on any value in $(0, 1]$. The rates of convergence in (3.3)–(3.5) are functions of the rate of convergence in (3.1). Specifically, without assuming more than $|\log t| - \rho x = o(x)$ in (3.1), the same $o(x)$ term appears in the convergence of $K(x, t)$, $|\log(1 - H(x, t))|$, and $|\log(1 - P(x, t))|$ in (3.3)–(3.5). However, when $\rho > 1$, the convergence is $O(1/x)$ in (3.3) and (3.4), but in no other cases, an artifact of the natural upper bound $K(x, t) \leq x$ that is relevant only in the $\rho > 1$ case.

The result (3.3) can equivalently be stated as, under (3.1),

$$K(x, t) \sim \frac{x}{j} + \left(\frac{j-1}{2}\right) |\log t| \quad (3.6)$$

as $t \rightarrow 0$ for j satisfying (3.2). Hence, for small t , the first quadrant of the (x, t) -plane can be thought of as partitioned into the regions

$$R_j = \left\{ (x, t) : x > 0, t > 0, \left(\frac{j+1}{2}\right)^{-1} \leq \frac{|\log t|}{x} < \left(\frac{j}{2}\right)^{-1} \right\}, \quad j = 1, 2, \dots, \quad (3.7)$$

which determine the asymptotic behavior of the optimal strategy. Figure 1 plots (3.6) and the boundaries of the first few R_j . Note that although (3.6) varies smoothly within each R_j , it is continuous but not smooth at the lower boundary of R_j . For small t , $K(x, t)$ given by (3.3) turns out to be such that if $(x, t) \in R_j$, then after firing $K(x, t)$ at an immediate enemy, the new state $(x - K(x, t), t)$ lies in R_{j-1} . This leads to the inductive method of proof, given in the next section. The boundary of the R_1 region is asymptotically equivalent to the estimates of the spend-it-all region's boundary in Theorem 2.1 in the strong sense that their difference is $o(1)$ as $t \rightarrow 0$.

3.1. Proof of Theorem 3.1

We proceed by induction on j . To begin, assume that $j = 1$. The optimal conditional survival probability $H(x, t)$ is bounded below by the conditional survival probability

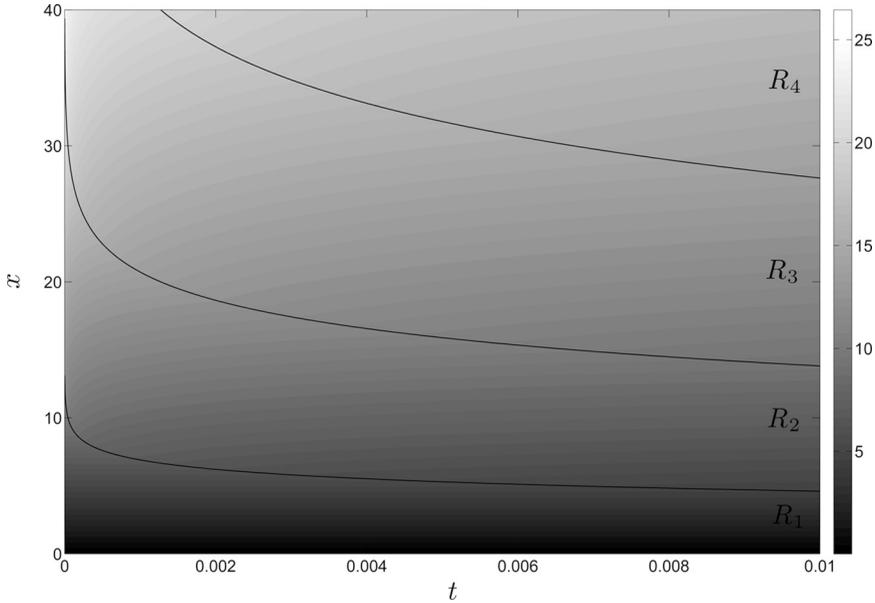


Figure 1. The small- t asymptotic approximation (3.6) of $K(x, t)$.

of the spend-it-all strategy (2.11), giving

$$\begin{aligned}
 H(x, t) &\geq a(x)e^{-tu} \\
 &\geq (1 - ve^{-x})(1 - tv) \\
 &= 1 - ve^{-x} - ve^{-\rho x + o(x)} + v^2 e^{-(\rho+1)x + o(x)}.
 \end{aligned} \tag{3.8}$$

Moving to $P(x, t)$, it is bounded below by the survival probability of the strategy that fires x at the first enemy. Under such a strategy, the bomber survives if no enemy plane arrives during time t , which happens with probability e^{-t} , or if the bomber encounters and survives one enemy, which happens with probability $te^{-t}a(x)$, and ignoring other enemy encounters we obtain

$$\begin{aligned}
 P(x, t) &\geq e^{-t}[1 + ta(x)] \\
 &\geq (1 - t)[1 + t - vte^{-x}] \\
 &= 1 - vte^{-x} - t^2 + vt^2e^{-x} \\
 &= 1 - ve^{-(\rho+1)x + o(x)} - e^{-2\rho x + o(x)} + ve^{-(2\rho+1)x + o(x)} \\
 &\geq 1 - ve^{-(\rho+1)x + o(x)}.
 \end{aligned} \tag{3.9}$$

On the other hand, $P(x, t)$ is bounded above by the survival probability of the infeasible strategy that fires x at the first enemy and, upon survival of this encounter, is guaranteed survival thereafter, so

$$\begin{aligned}
 P(x, t) &\leq e^{-t}[1 + a(x)(e^t - 1)] \\
 &= 1 - ve^{-x} + ve^{-x}e^{-t}
 \end{aligned}$$

$$\begin{aligned}
&\leq 1 - ve^{-x} + ve^{-x}(1 - t + t^2/2) \\
&= 1 - vte^{-x} + vt^2e^{-x}/2 \\
&= 1 - ve^{-(\rho+1)x+o(x)} + ve^{-(2\rho+1)x+o(x)}/2 \\
&= 1 - e^{-(\rho+1)x+o(x)}. \tag{3.10}
\end{aligned}$$

For this $j = 1$ case, we consider separately the cases $\rho = 1$ and $\rho \in (1, \infty)$. First assume that $\rho > 1$. In this case, (3.8) is $1 - ve^{-x}(1 + o(1))$, and as the conditional probability of the spend-it-all strategy is bounded above by $a(K(x, t)) \cdot 1$, the probability of surviving the first encounter when expending the optimal amount $K(x, t)$ and ignoring future danger, we find that $1 - ve^{-x}(1 + o(1)) \leq 1 - ve^{-K(x, t)}$ so that $x + o(1) \leq K(x, t) < x$. To estimate $H(x, t)$, plug $K(x, t) = x + o(1)$ into $a(K(x, t))$ to get

$$a(K(x, t)) = 1 - ve^{-x}(1 + o(1)).$$

This is the same order as the lower bound (3.8), hence $H(x, t) = 1 - ve^{-x}(1 + o(1))$, which implies (3.4) in this case. The limit (3.5) holds as well in this case since both (3.9) and (3.10) are of order

$$1 - e^{-(\rho+1)x+o(x)}.$$

Since $H(x, t) = 1 - ve^{-x}(1 + o(1))$ is equivalent to

$$\frac{1}{x} |\log(1 - H(x, t))| = 1 - (\log v)/x + o(1/x) = 1 + O(1/x),$$

the error term on the right hand side of (3.4) in this case is $O(1/x)$; this holds for (3.3) and (3.4) when $\rho > 1$, but for no other cases.

Now let $\rho = 1$. The lower bound (3.8) is

$$1 - 2ve^{-x+o(x)} = 1 - e^{-x+o(x)} \tag{3.11}$$

and by Lemma 3.2 below we have $x + o(x) \leq K(x, t) \leq x$. Plugging $K(x, t) = x + o(x)$ into the upper bound $a(K(x, t))$ gives $H(x, t) \leq 1 - e^{-x+o(x)}$ the same order as the lower bound (3.11), hence

$$\frac{1}{x} |\log(1 - H(x, t))| \rightarrow 1.$$

The lower bound (3.9) gives

$$P(x, t) \geq 1 - e^{-2x+o(x)},$$

and the upper bound (3.10) gives the same order, hence

$$\frac{1}{x} |\log(1 - P(x, t))| \rightarrow 2 = \rho + 1.$$

This concludes the $j = 1$ case.

Now let I_j denote the half-closed interval (3.2), i.e.,

$$I_j = \left[\binom{j+1}{2}^{-1}, \binom{j}{2}^{-1} \right), \quad (3.12)$$

and let $\alpha_j(\rho)$ and $\beta_j(\rho)$ denote the right-hand sides of (3.3) and (3.5), respectively, i.e.,

$$\alpha_j(\rho) = 1/j + \rho(j-1)/2 \quad (3.13)$$

$$\beta_j(\rho) = 1/j + \rho(j+1)/2 \quad (3.14)$$

For the inductive step, assume that (3.3)–(3.5) hold for j and let ρ belong to I_{j+1} . $H(x, t)$ is bounded below by the conditional survival probability of the strategy $\underline{H}(x, t)$ that fires $\tilde{K}(x) = \alpha_{j+1}(\rho)x$ at the first enemy, and then behaves optimally thereafter. Letting

$$x' := x - \tilde{K}(x) = x[1 - \alpha_{j+1}(\rho)],$$

we have

$$\rho' := \lim_{t \rightarrow 0} \frac{|\log t|}{x'} = \frac{\rho}{1 - \alpha_{j+1}(\rho)} \in I_j$$

by Lemma 3.1 below. Then, by the inductive hypothesis, we have

$$\underline{H}(x, t) = a(\tilde{K}(x))P(x', t) = [1 - ve^{-\alpha_{j+1}(\rho)x}][1 - e^{-\beta_j(\rho')x' + o(x)}], \quad (3.15)$$

and

$$\beta_j(\rho') \frac{x'}{x} = \beta_j(\rho')[1 - \alpha_{j+1}(\rho)] = \alpha_{j+1}(\rho)$$

by Lemma 3.1, giving

$$\underline{H}(x, t) = [1 - e^{-\alpha_{j+1}(\rho)x + o(x)}]^2 = 1 - e^{-\alpha_{j+1}(\rho)x + o(x)} \quad (3.16)$$

Lemma 3.2 then implies that

$$K(x, t) \geq \alpha_{j+1}(\rho)x + o(x), \quad (3.17)$$

and we will show that this expression actually holds with equality. To do this, we consider sequences (x, t) still for which $|\log t|/x \rightarrow \rho \in I_{j+1}$ and on which

$$\tau := \lim_{t \rightarrow 0} \frac{K(x, t)}{x}$$

exists, and we will show that $\tau = \alpha_{j+1}(\rho)$ is the only possible limit. This suffices to show that the lim sup and lim inf of $K(x, t)/x$ both equal $\alpha_{j+1}(\rho)$.

By (3.17), we know that the only possible values of τ lie in $[\alpha_{j+1}(\rho), 1]$. First, suppose that there is a sequence (x, t) on which $\tau \in (\alpha_{j+1}(\rho), 1)$. Then

$$x'' := x - K(x, t) \sim (1 - \tau)x \quad \text{and} \quad \rho'' := \lim_{t \rightarrow 0} \frac{|\log t|}{x''} = \frac{\rho}{1 - \tau} > \frac{\rho}{1 - \alpha_{j+1}(\rho)} = \rho' \in I_j$$

by Lemma 3.1, so let $i \in \{1, 2, \dots, j\}$ be such that $\rho'' \in I_i$. Then, again by the inductive hypothesis, we would have

$$H(x, t) = a(K(x, t))P(x'', t) = [1 - ve^{-\tau x + o(x)}][1 - e^{-\beta_i(\rho'')x'' + o(x'')}] \tag{3.18}$$

and

$$\begin{aligned} \beta_i(\rho'') \frac{x''}{x} &= \left[\frac{1}{i} + \frac{\rho''(i+1)}{2} \right] \frac{x''}{x} \rightarrow \left[\frac{1}{i} + \frac{\rho(i+1)}{2(1-\tau)} \right] (1-\tau) \\ &= \frac{1-\tau}{i} + \frac{\rho(i+1)}{2}. \end{aligned} \tag{3.19}$$

If $i < j$, then $\rho/(1-\tau) \in I_i$ implies that $(1-\tau) \leq \rho \binom{i+1}{2}$, so (3.19) becomes

$$\begin{aligned} \frac{1-\tau}{i} + \frac{\rho(i+1)}{2} &\leq \rho(i+1) \\ &= \rho(i+1 - j/2) + \rho j/2 \\ &< \left(\frac{j+1}{2} \right)^{-1} (j - j/2) + \rho j/2 \quad (\text{since } \rho \in I_{j+1} \text{ and } i < j) \\ &= \alpha_{j+1}(\rho). \end{aligned}$$

If $i = j$, then (3.19) becomes

$$\begin{aligned} \frac{1-\tau}{j} + \frac{\rho(j+1)}{2} &< \frac{1 - \alpha_{j+1}(\rho)}{j} + \frac{\rho(j+1)}{2} \\ &= \alpha_{j+1}(\rho) - \frac{[(j+1)\alpha_{j+1}(\rho) - 1]}{j} + \frac{\rho(j+1)}{2} \\ &= \alpha_{j+1}(\rho). \end{aligned}$$

In both cases we have shown that (3.19) is less than $\alpha_{j+1}(\rho) < \tau$, which implies that (3.18) is

$$1 - \exp[-((1 - \tau)/i + \rho(i + 1)/2)x + o(x)]$$

and is hence smaller than (3.16) for small t , a contradiction.

Now assume that there is a sequence (x, t) on which $\tau = 1$. Using the crude bound $a(K(x, t)) \leq 1$ and (2.9),

$$\begin{aligned} H(x, t) &= a(K(x, t))P(x - K(x, t), t) \\ &\leq 1 \cdot \exp[-tve^{-(x-K(x,t))}] \\ &\leq 1 - vte^{-(x-K(x,t))} + v^2t^2e^{-2(x-K(x,t))}/2 \\ &= 1 - ve^{-\rho x + o(x)} + v^2e^{-2\rho x + o(x)} \\ &= 1 - e^{-\rho x + o(x)}, \end{aligned}$$

which leads to the same contradiction since $\rho < \alpha_{j+1}(\rho)$ by Lemma 3.1. We have shown that $\alpha_{j+1}(\rho)$ is the only possible value of τ , hence (3.17) holds with equality and (3.16) holds for $H(x, t)$. All that remains is to verify (3.5) for the $j + 1$ case.

Let T denote the exponentially distributed waiting time, with mean 1, until the first enemy, and recall that we write $x = x_t$ to emphasize the dependence on t . Then P and H are related through the expectation

$$\begin{aligned} P(x, t) &= P(x_t, t) = E[H(x_t, t - T)\mathbf{1}\{T < t\} + \mathbf{1}\{T \geq t\}] \\ &= \int_0^t H(x_t, t - r)e^{-r} dr + P(T \geq t) \\ &= e^{-t} \left[\int_0^t H(x_t, s)e^s ds + 1 \right]. \end{aligned} \tag{3.20}$$

Using (3.20) and that $H(x, \cdot)$ is nonincreasing, we have

$$\begin{aligned} P(x, t) &\geq e^{-t} \left[H(x_t, t) \int_0^t e^s ds + 1 \right] \\ &= e^{-t} [H(x_t, t)(e^t - 1) + 1] \\ &= e^{-t} [1 - H(x_t, t)] + H(x_t, t) \\ &\geq (1 - t)[1 - H(x_t, t)] + H(x_t, t) \\ &= 1 - t[1 - H(x_t, t)] \\ &= 1 - e^{-\rho x + o(x)} e^{-\alpha_{j+1}(\rho)x + o(x)} \\ &= 1 - e^{-[\rho + \alpha_{j+1}(\rho)]x + o(x)} \\ &= 1 - e^{-\beta_{j+1}(\rho)x + o(x)} \end{aligned}$$

by Lemma 3.1. We bound $P(x, t)$ from above by a function of the same order. Fix $\delta \in (0, 1)$ and note that

$$\frac{|\log(\delta t)|}{x_t} = \frac{-\log(\delta t)}{x_t} = \frac{-\log t}{x_t} + \frac{-\log \delta}{x_t} = \frac{|\log t|}{x_t} + \frac{|\log \delta|}{x_t} \rightarrow \rho. \tag{3.21}$$

Then, by (3.20),

$$\begin{aligned} P(x, t) &\leq e^{-t} \left[\int_0^{\delta t} e^s ds + H(x_t, \delta t) \int_{\delta t}^t e^s ds + 1 \right] \\ &= e^{-(1-\delta)t} [1 - H(x_t, \delta t)] + H(x_t, \delta t) \\ &\leq [1 - (1 - \delta)t + t^2][1 - H(x_t, \delta t)] + H(x_t, \delta t) \\ &= 1 - (1 - \delta)t[1 - H(x_t, \delta t)] + t^2[1 - H(x_t, \delta t)] \\ &= 1 - (1 - \delta)e^{-\rho x + o(x)} e^{-\alpha_{j+1}(\rho)x + o(x)} + e^{-2\rho x + o(x)} e^{-\alpha_{j+1}(\rho)x + o(x)} \quad (\text{by (3.21)}) \\ &= 1 - e^{-[\rho + \alpha_{j+1}(\rho)]x + o(x)} \\ &= 1 - e^{-\beta_{j+1}(\rho)x + o(x)}, \end{aligned}$$

completing the proof of Theorem 3.1, except for the following lemmas. The first collects various facts relating $\alpha_j(\rho)$, $\beta_j(\rho)$, and ρ , and the second provides a crude but useful bound on $K(x, t)$.

Lemma 3.1. Let I_j , $\alpha_j(\rho)$, and $\beta_j(\rho)$ be as in (3.12)–(3.14). Assume that $\rho \in I_{j+1}$ for some $j \geq 1$, and let $\rho' = \rho/[1 - \alpha_{j+1}(\rho)]$. Then

$$\rho < \alpha_{j+1}(\rho), \quad (3.22)$$

$$\rho' \in I_j, \quad (3.23)$$

$$\beta_j(\rho') = [1/\alpha_{j+1}(\rho) - 1]^{-1}, \quad (3.24)$$

$$\alpha_{j+1}(\rho) + \rho = \beta_{j+1}(\rho). \quad (3.25)$$

Proof. Let $\rho \in I_{j+1}$. Then

$$\begin{aligned} \beta_j(\rho') &= \frac{1}{j} + \frac{\rho'(j+1)}{2} \\ &= \frac{1}{j} + \frac{2(j+1)^2}{2j(2/\rho - (j+1))} \\ &= \frac{\alpha_{j+1}(\rho)}{1 - \alpha_{j+1}(\rho)} \end{aligned}$$

after some simplifying, proving (3.24). For (3.22),

$$\begin{aligned} \rho &= \rho(1 - j/2) + \rho j/2 \\ &\leq \begin{cases} \rho/2 + \rho/2, & j = 1 \\ \rho j/2, & j \geq 2 \end{cases} \\ &< \begin{cases} 1/2 + \rho/2, & j = 1 \\ 1/(j+1) + \rho j/2, & j \geq 2 \end{cases} \\ &= \alpha_{j+1}(\rho). \end{aligned}$$

For (3.23),

$$\begin{aligned} \rho' &= \frac{2(j+1)}{j[2/\rho - (j+1)]} \\ &\in \left[\frac{2(j+1)}{j[2\binom{j+2}{2} - (j+1)]}, \frac{2(j+1)}{j[2\binom{j+1}{2} - (j+1)]} \right) \\ &= \left[\binom{j+1}{2}^{-1} \binom{j}{2}^{-1} \right) = I_j. \end{aligned}$$

For (3.25), $\alpha_{j+1}(\rho) + \rho = 1/(j+1) + \rho(j+2)/2 = \beta_{j+1}(\rho)$. \square

Lemma 3.2. If there is a $j \in (0, 1]$ such that $H(x, t) \geq 1 - e^{-\gamma x + o(x)}$, then $K(x, t) \geq \gamma x + o(x)$.

Proof. We have

$$H(x, t) = a(K(x, t))P(x - K(x, t), t) \leq a(K(x, t)) \cdot 1 = 1 - ve^{-K(x, t)},$$

and setting this last \geq the assumed lower bound $1 - e^{-\gamma x + o(x)}$ leads to $K(x, t) \geq \gamma x + o(x)$. \square

4. DISCUSSION

In Section 3 an inductive method is used to estimate the limiting optimal fraction $K(x, t)/x$ of ammunition used as $t \rightarrow 0$. The same result holds when the bomber is restricted to only firing discrete units (integers, say) of ammunition x , the only modification of the proof needed is to replace x by $\lfloor x \rfloor$ (the largest integer $\leq x$) in the appropriate places. For example, in the $\rho > 1$ case in the proof of Theorem 3.1, we have $H(x, t) \geq a(\lfloor x \rfloor)e^{tv}$, which leads to $\lfloor x \rfloor + o(1) \leq K(x, t) \leq \lfloor x \rfloor$, and hence $K(x, t)/x \rightarrow 1$, using that $\lfloor x \rfloor/x \rightarrow 1$.

Theorem 2.1 shows that $K(x, t) = x$ in a region asymptotically equivalent to R_1 in (3.7) and, this being monotone in x , that conjecture [B] holds in this region. It is therefore natural to ask if the estimates of $K(x, t)$ in R_j given by Theorem 3.1 can be used to shed any light on conjecture [B] for $j \geq 2$. One thing we can say is that [B] is satisfied *in the limit* as $t \rightarrow 0$ in the following sense. Letting $x_1 \leq x_2$ be such that $\lim_{t \rightarrow 0} |\log t|/x_1 \in R_j$ and $\lim_{t \rightarrow 0} |\log t|/x_2 \in R_j$ for some $j \leq j'$, by (3.6) we have

$$K(x_2, t) - K(x_1, t) \sim \frac{x_2}{j'} + \left(\frac{j' - 1}{2}\right) |\log t| - \left[\frac{x_1}{j} + \left(\frac{j - 1}{2}\right) |\log t|\right]. \quad (4.1)$$

If $j = j'$, then (4.1) is $(x_2 - x_1)/j \geq 0$. If $j < j'$, then (4.1) divided by $|\log t|$ is

$$\begin{aligned} \frac{x_2}{j'|\log t|} - \frac{x_1}{j|\log t|} + \frac{j' - j}{2} &> \left(\frac{\binom{j'}{2}}{j'} - \frac{\binom{j+1}{2}}{j} + \frac{j' - j}{2}\right)(1 + o(1)) \\ &= (j' - j - 1)(1 + o(1)) \end{aligned}$$

which approaches a nonnegative limit. However, to make this argument hold for, say, all $x_1 \leq x_2$ sufficiently large and all t sufficiently small, higher order asymptotics are needed. In particular, the rate of convergence in (3.3) as a function of x and t is needed, for which the tools developed in Sections 2 and 3 may be a starting point.

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